Sensing and Perception:
Localization and positioning
Isaac Skog

Homework: GNSS-positioning

In this homework you will study how global navigation satellite system (GNSS) receiver calculates its position estimates. The GNSS-receiver is going to be used to track the motion of car driving around a race track and your task is to: (i) implement a nonlinear least squares estimator that estimates the position of the car, and (ii) study the effect of range errors and the satellite geometry constellation on the accuarcy of the GNSS-receiver. On the course homepage you can download a matlab file `GPSdata.mat` with the true (position) trajectory of the car and the data recorded by the GNSS-receiver.

Today the global positioning system (GPS), which is one out of four GNSS systems, consists of 30 satellites that orbits the earth at an altitude of 22 000 km. These satellites constantly transmits time encoded signals that may be received by a GPS-receiver at earth. The GPS-receiver uses the received signals to calculated the distances between the GPS-receiver and the satellites. These distance estimates are generally referred to as pseudo-range measurements since they also include an range offset caused by the offset between the GPS-receiver clock and the GPS satellite clocks. The pseudo range measurement to the i:th satellite at time k can be modelled as

$$y_k^{(i)} = h_i(\mathbf{p}_k^{(rec)}, \Delta t_k) + v_k^{(i)}, \tag{1}$$

where

$$h_i(\mathbf{p}, \Delta t_k) = \|\mathbf{p}_k^{(i)} - \mathbf{p}\| + c\,\Delta t_k. \tag{2}$$

Here $\mathbf{p}_k^{(i)}$ and $\mathbf{p}_k^{(rec)}$ denotes the position of the i:th satellite[1] and the receiver, respectively. Further, c denotes the speed of light, Δt_k denotes the offset in the GPS-receivers clock, and $v_k^{(i)}$ denotes the measurement noise, which can be assumed zero-mean with the covariance $\mathbb{E}\{v_k^{(i)} v_l^{(j)}\} = \sigma_{r_i}^2\,\delta_{i-j,k-l}$. Assuming that signals from M satellites are received, then the measurements from these satellites can be modelled as

$$\mathbf{y}_k = \mathbf{h}(\boldsymbol{\theta}_k) + \mathbf{v}_k \tag{3}$$

where

$$\mathbf{y}_k = \begin{bmatrix} y_k^{(1)} & \dots & y_k^{(M)} \end{bmatrix}^\top \quad \boldsymbol{\theta}_k = \begin{bmatrix} (\mathbf{p}_k^{(rec)})^\top & \Delta t_k \end{bmatrix}^\top \quad \mathbf{v}_k = \begin{bmatrix} v_k^{(1)} & \dots & v_k^{(M)} \end{bmatrix}^\top$$

and

$$\mathbf{h}(\boldsymbol{\theta}_k) = \begin{bmatrix} h_1(\mathbf{p}_k^{(rec)}, \Delta t_k) & \dots & h_M(\mathbf{p}_k^{(rec)}, \Delta t_k) \end{bmatrix}^\top.$$

[1] The position of the satellite $\mathbf{p}_k^{(i)}$ is known by the receiver, but the satellites from which the GPS-receiver can receive signals varies with time and the position of the receiver.

Given the signal model in (3) the nonlinear least square estimator for the position and clock offset is given by

$$\widehat{\boldsymbol{\theta}}_k = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\Big(\|\mathbf{y}_k - \mathbf{h}(\boldsymbol{\theta})\|^2_{\mathbf{Q}^{-1}} \Big) \tag{4}$$

where $\|\mathbf{a}\|^2_{\mathbf{P}} = \mathbf{a}^\top \mathbf{P}\mathbf{a}$ and $\mathbf{Q} = \operatorname{diag}(\sigma^2_{r_1}, \ldots, \sigma^2_{r_M})$.

TASK #1: Download the material from the home page and implement a nonlinear least square estimator that estimates the positions of the car. You may use Matlab's nonlinear least squares solvers or implement you own solver. Print a plot that shows the estimated trajectory as well as the true trajectory of the car.

TASK #2: What is the minimum number of satellites M that the GNSS-receiver must be able to measure the distance to in order for the nonlinear least square estimation problem in (4) to be well defined?

The accuracy of the nonlinear least square estimator is given by

$$\operatorname{Cov}(\hat{\boldsymbol{\theta}}) \simeq \Big(\mathbf{H}^\top \mathbf{Q}^{-1} \mathbf{H} \Big)^{-1} \tag{5}$$

where

$$\mathbf{H} = \begin{bmatrix} \nabla_{\boldsymbol{\theta}} h_1 \\ \vdots \\ \nabla_{\boldsymbol{\theta}} h_M \end{bmatrix} \tag{6}$$

and $\nabla_{\boldsymbol{\theta}} h_i$ is the gradient of the function $h_i(\mathbf{p}, \Delta t_k)$ defined in (2).

TASK #3: Use (5) to evaluate the theoretical accuracy of the nonlinear least squares solver that you implemented in Task #1. Plot the 3σ bounds for the horizontal and vertical accuracy as well as the horizontal and vertical estimation error as a function of time. How is the agreement between the theoretical and true accuracy? In which direction is the positioning accuracy worst?

TASK #4: Assuming that only five satellites are used in the position calculation, and the satellites positions in the local tangent plane (North,East, Up) in polar coordinates {Azimuth, Inclination} are given by $\{0, \pi/2\}$, $\{0, x\}$, $\{\pi/2, x\}$,$\{\pi, x\}$, and $\{3\pi/4, x\}$. Further, assume that variance of the ranging error are the same for all satellites. Plot how the horizontal and vertical accuracy varies with the inclination $x \in (0, \pi/2)$. By aid of the plots, discuss how an urban environment with a lot of high-rise building effects the positioning accuracy of the GPS receiver. What other reasons are there that may cause a GPS receiver to work poorly in an urban area with a lot of high rise buildings?

Unfortunately, most GPS receivers doesn't output the full covariance matrix $\text{Cov}(\hat{\boldsymbol{\theta}})$, but instead output one, or multiple, so called dilution-of-precision (DOP) figures. These DOP figures indicates the goodness of the satellite geometry from a positioning accuracy perspective, and can be related to the position accuracy as

$$\text{Tr}\{\text{Cov}(\hat{\boldsymbol{\theta}})\} \approx \text{UERE} \cdot \text{DOP}. \tag{7}$$

Here, UERE stands for user equivalent ranging error and is a measure of the size of the typical ranging error.

TASK #5: If $\sigma_{r_i}^2 = \sigma_{r_j}^2 \ \forall i, j$, show that the right hand side of (5) can be factorized in two parts, one which only depends on the ranging accuracy and one that only depends on the direction to the satellites from the GNSS-receiver.

TASK #6: If the GPS-receiver position estimates are to be fused with the measurements from another positioning sensor, what is the drawback of only having access to the dilution-of-precision (DOP) figures and not the full covariance matrix $\text{Cov}(\hat{\boldsymbol{\theta}})$?